Optimal Controller Design for Inverted Pendulum System: An Experimental Study

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Optimal Controller Design for Inverted Pendulum System: An Experimental Study

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CERTIFICATE

*This is to certify that the thesis titled "**Optimal Controller Design for Inverted Pendulum: An Experimental Study**", by **Prasanna Priyadarshi**, submitted to the National Institute of Technology, Rourkela for the award of degree of Master of Technology with specialization in **Control & Automation** is a record of bona fide research work carried out by him in the Department of Electrical Engineering, under my supervision. I believe that this thesis fulfills part of the requirements for the award of degree of Master of Technology. The results embodied in this thesis have not been submitted in parts or full to any other University or Institute for the award of any other degree elsewhere to the best of my knowledge.*

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Dedicated to

My Respected Nanaji & Nani Maa

My Wonderful Maa and Papa

My beloved Bhaiya (Anand Priyadarshi)

My Younger Sister (Kumari Himshweta)

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ABSTRACT

The Cart Inverted Pendulum System (CIPS) has been considered among the most classical and difficult problem in the field of control engineering. The Inverted Pendulum is considered among the typical representative of a class of under actuated, non-minimal system with non-linear dynamics.

The aim of this study is to stabilize the Inverted Pendulum such that position of the cart on 1 meter track is controlled quickly and accurately so that pendulum is always maintained erected in its upright (inverted) position.

This thesis begins with the explanation of CIPS together with the hardware setup used for research, its state space dynamics and transfer function models after linearizing it. Since, Inverted Pendulum is inherently unstable i.e. if it is left without a stabilizing controller it will not be able to remain in an upright position when disturbed. So, a systematic iterative method for the state feedback design by choosing weighting matrices key to Linear Quadratic Regulator (LQR) design is presented assuming all the states to be available at the output. After that, Kalman Filter, which is an optimal Observer has been designed to estimate all the four states considering process and measurement noises in the system.

Then, a Full State Feedback Controller i.e. Linear Quadratic Gaussian (LQG) compensator has been designed. The compensator aims at providing a proper control input that provides a desired output in terms of the Pendulum Angle and Cart Position. Simulation and Experimental study has been carried out to demonstrate the effectiveness of the proposed approach in meeting the desired specifications. Lastly, Loop Transfer Recovery (LTR) analysis has been performed depending on the trade-off between noise suppression and system robustness for suitably selecting the tuning parameter for Observer design.

CONTENTS

LIST OF ABBREVIATIONS

Abbreviation	Description
IFAC	International Federation of Automatic Control
CIPS	Cart-Inverted Pendulum System
SIMO	Single-Input-Multi-Output
DC	Direct Current
LQR	Linear Quadratic Regulator
LQG	Linear Quadratic Regulator
LTR	Loop Transfer Recovery
PID	Proportional Integral Derivative
DOF	Degrees Of Freedom
FBD	Free Body Diagram
A/D	Analog-to-Digital
PD	Proportional-Derivative
PI	Performance Index
CF	Cost Functional
ARE	Algebraic Riccati Equation

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INTRODUCTION

An Inverted Pendulum is a popular mechatronic application that exists in different form. Balancing control of Inverted Pendulum system has attracted the attention of both Researchers and educators and has many applications such as walking control of Humanoid Robot.

The International Federation of Automatic Control (IFAC) Theory Committee in the year 1990 has determined a set of practical design problems that are helpful in comparing new and existing control methods and tools so that a meaningful comparison can be derived. The committee came up with a set of real world control problems that were included as “benchmark control problems”. Out of which the cascade inverted pendulum control problem is featured as highly unstable, and the toughness increases with increase in the number of links. Anderson and Pandy (2003) reported briefly on the dynamics of the inverted pendulum as a model of stance phase and Buckzek and his team in more detail (2006).

The Inverted Pendulum is a classical control problem in dynamics and control theory and is widely used as a benchmark for testing control algorithm (PID controller, neural network, fuzzy control, genetic algorithm etc.). The simplest case of this system is the cart- single inverted pendulum system. It also has very good practical applications right from missile launchers to segways, human walking, luggage carrying pendubots, earthquake resistant building design etc. The Inverted Pendulum dynamics resembles the missile or rocket launcher dynamics as its center of gravity is located behind the centre of drag causing aerodynamic instability.

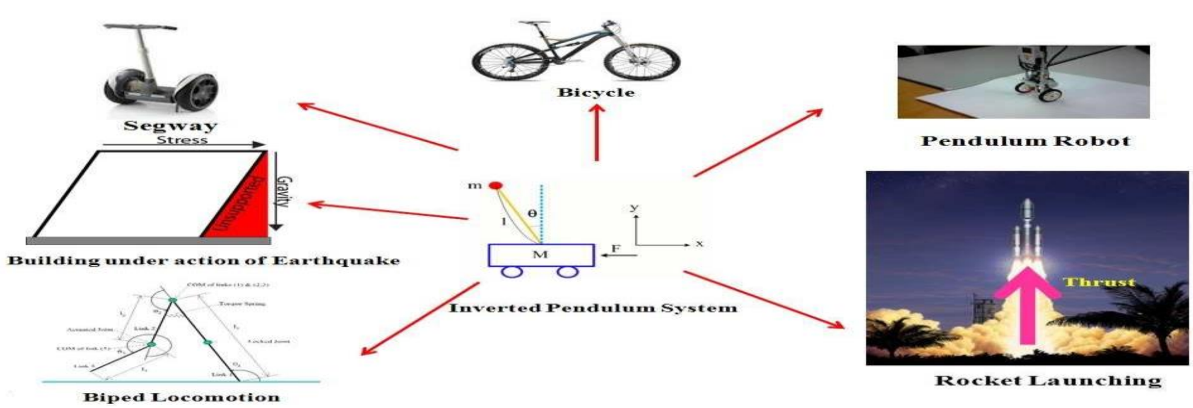


Figure 1.1.Various Applications of Inverted Pendulum like systems

Inverted pendulum is among the most difficult systems to control in the field of control engineering due to its importance in the field of control engineering, it has been a task of choice to be assigned to control engineering students to analyze the model.

The reasons for selecting the Inverted Pendulum (IP) as the system are:-

- It is the most easily available system (in most academia) for laboratory usage.
- It is a nonlinear system, which can be treated to be linear, without much error, for quite a wide range of variation.
- Provides a good practice for prospective control engineers.

1.1. Introduction to Inverted Pendulum Control Problem

The Inverted Pendulum, a highly Non-Linear and unstable system is very common control problem being assigned to a student of control system engineering. It is used as a benchmark for implementing the control methods. The problem is referred in classical literature as *pole balancer control problem, cart-pole problem, broom balancer control problem, stick balancer control problem, inverted pendulum control problem.* The Inverted Pendulum setup consist of a D.C. Motor, a pendant type pendulum, a cart, and a driving mechanism. Fig.1.2.shows the basic schematic diagram for the cart-inverted pendulum system:-

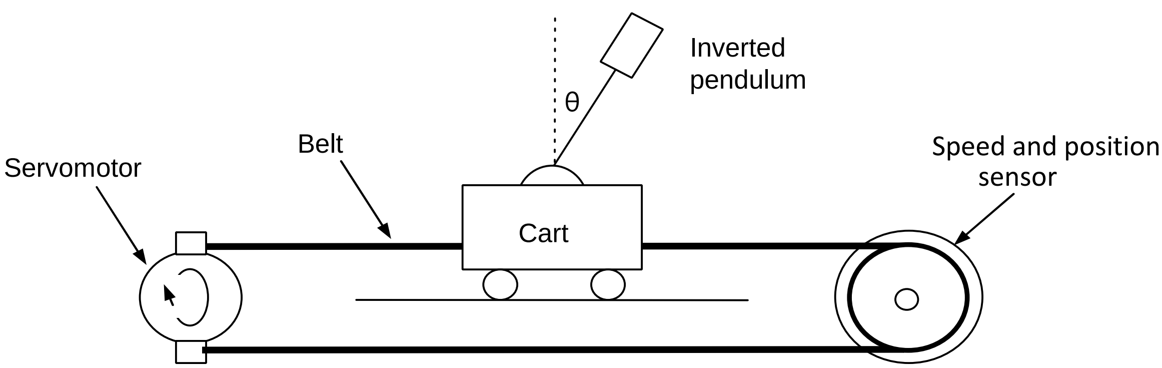


Figure 1.2. Schematic diagram of cart Inverted Pendulum system

There are basically two kind of inverted pendulum control problems. First one is based on the rocking of pendulum base point to keep it upright. The second one is to control the moving base point so as to get the pendulum stable in upright position. The lab experimental set-up is based on the second approach.

The Inverted Pendulum System is Single Input Multiple Output (SIMO) type of system. Here, there are two no. of free component i.e. it has 2 degree of freedom. It has one input i.e. D.C. voltage and two variables that are read from the pendulum using optical encoders as outputs are position of cart, x and angle of pendulum, θ. The inverted pendulum is a challenging control problem due to the various characteristics of the system:-

- **Highly Nonlinear -** The dynamic equations of the CIPS consists of non-linear terms.

- **Highly Unstable-** The inverted position is the point of unstable equilibrium as can be seen from the non-linear dynamic equations.

- **Non-Minimum Phase System-** The system transfer function of CIPS contains right hand plane zeros, which affect the stability margins including the robustness.

- **Under- actuated Mechanical System-**The system has two degrees of freedom of motion but only one actuator i.e. the D.C. Motor. Thus, this system is under-actuated. This makes the system cost effective but the control problem becomes challenging.

1.2. Application of Inverted Pendulum

Some of the considerable applications of Inverted Pendulum (IP) are:

1.2.1. Simulation of Dynamics of a Robotic Arm

The Inverted Pendulum problem resembles the control systems that exist in robotic arms. The dynamics of Inverted Pendulum simulates the dynamics of robotic arm in the condition when the center of pressure lies below the center of gravity for the arm so that the system is also unstable. Robotic arm behaves very much like Inverted Pendulum under this condition.

1.2.2. Model of a Human Standing Still

The ability to maintain stability while standing straight is of great importance for the daily activities of people. The central nervous system (CNS) registers the pose and changes in the pose of the human body, and activates muscles in order to maintain balance.

The inverted pendulum is widely accepted as an adequate model of a human standing still (quiet standing).

1.3. Experimental Setup Description

The Inverted Pendulum Experimental Set-up in laboratory consists of the following: - [13]

- PC with PCI-1711 card
- Digital Pendulum Controller
- Feedback SCSI Cable Adaptor
- Cart
- Track of 1m length with limit switches.
- DC Motor (Actuator)
- Pendant Pendulum with weight
- Optical encoders with HCTL2016 ICs
- Software: MATLAB, SIMULINK, Real-Time Workshop, ADVANTECH PCI-1711 device driver, Feedback Pendulum Software.
- Adjustable feet with belt tension adjustment.
- Connection cables and wires.

The heart of the experimental setup is a cart and a pendant pendulum. The cart has four wheels to slide on the track. There are two coupled pendant pendulums; they have a pendant or bob that would make the pendulum more unstable that is because it shifts the center of gravity to a higher level to the reference. The cart on the rail and is driven by a toothed belt which is driven by DC Motor. The motor drives the cart in a velocity proportional to the applied control voltage.

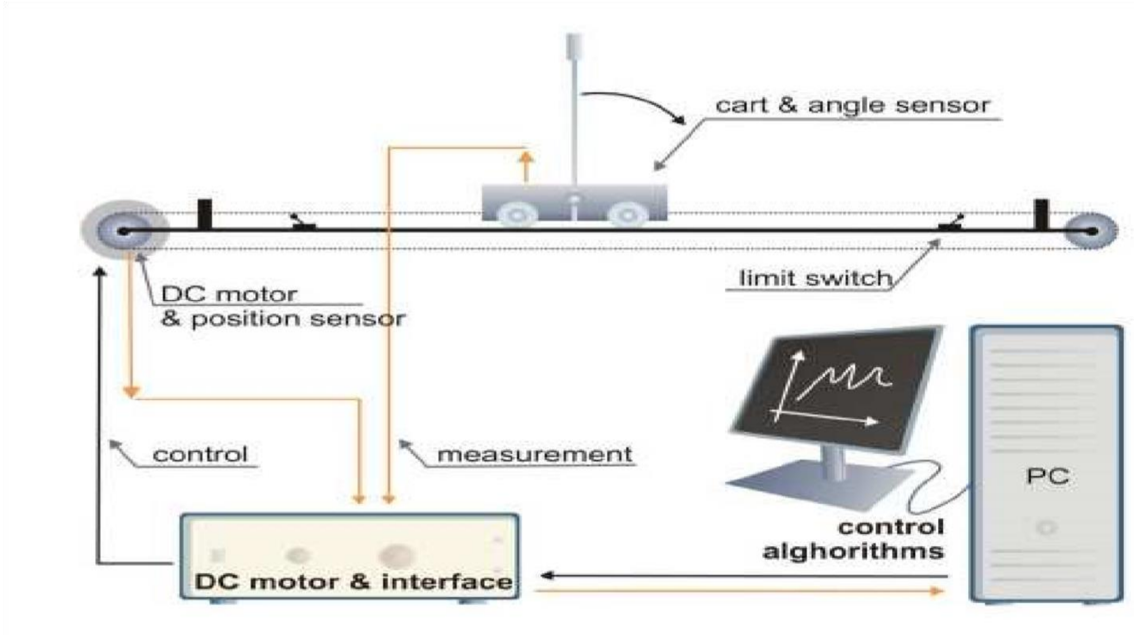


Fig.1.3.Feedback's Digital Pendulum Experimental Setup Diagram [14]

The motion of the cart is bounded mechanically and additionally for safety is improved by limit switches that cuts off power when the cart crosses them.

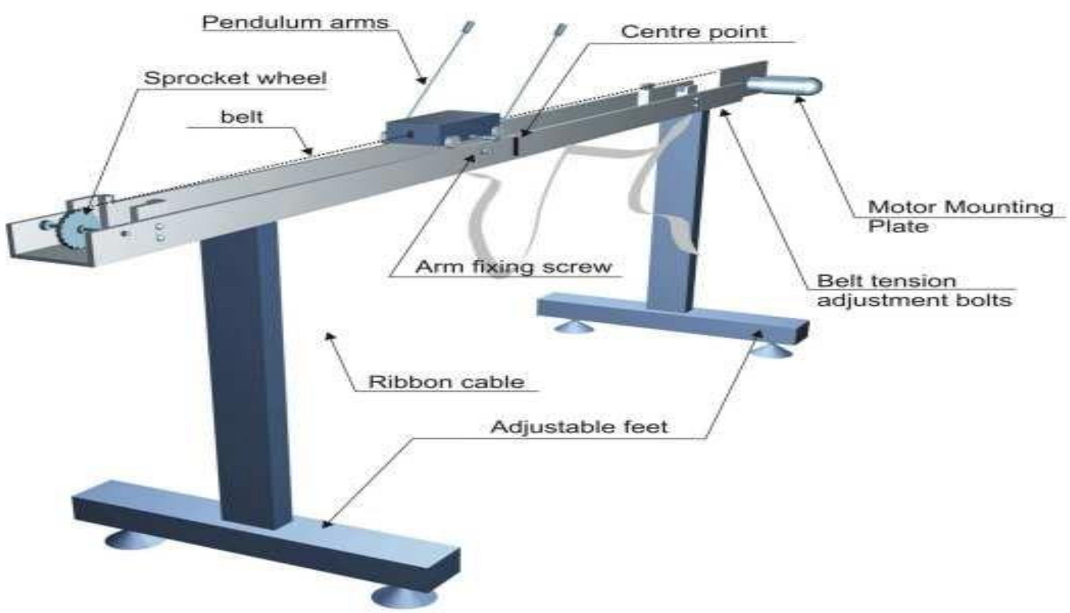


Fig.1.4.Digital Pendulum Mechanical Setup [14]

1.4. Literature Review: Control Strategies applied to Cart-Inverted Pendulum System

As more demanding characteristics are being required for mechanical system, better control of the system is also required. Furthermore, as system in the future becomes more complicated to perform more functions, in future engineers need to have a better understanding of control system & control theory.

The proposed Inverted Pendulum system fits the need. The Inverted Pendulum control problem is a solid starting point for testing different control algorithm on a physical system. The Inverted Pendulum system can further be complicated to test control algorithm on more complicated system.

The aim of this thesis is to stabilize the Inverted Pendulum (IP) such that position of the Cart on the track is controlled quickly and accurately so that the pendulum is always hold in this inverted position during such movements.

Control problems consists of obtaining the dynamic models of the system and by using this model to determine control laws to achieve the desired system response and performance [1].

Linear Quadratic Regulator(**LQR**) is an optimal control method which provides an alternative design strategy by which all the control design parameters can be determined even for Multi-Input, Multi-output system. It allows us to directly formulate the performance objectives of a control system [2] [21] [22]. It is one of the most widely used static state feedback methods. . It is equivalent to a two loop PD control design. In [3], stabilization of the cart pendulum system was carried out by linearization of the state model and designing a LQR after swing-up by an energy based controller.

There are two sets of poles one set is fast and other set is sluggish, the faster set of poles determine the angle dynamics and the slower set of poles determines the position dynamics. The cart position error always overshoots initially to catch up with the falling pendulum. Only after the rod is stabilized the position comes back to origin [4]. The effect of Inverted Pendulum under the linear state feedback has been analyzed in [5], the dynamic equations indicate the existence of stability regions in four dimensional state-space and an algorithm has been developed that transforms the four dimensional state space to three dimensional space. In [6], a tutorial has been presented wherein, the concept of digital control system design by pole placement with and without state estimation has been introduced.

LQG can be used in both the linear time-invariant system and as well as linear time-variant system. The application to linear time-variant system enables the design of linear feedback controller for non-linear uncertain systems, which is the case for the Cart Inverted pendulum system [8]. The **LQG** controller is simply the combination of **Kalman Filter** with that of **LQR** regulator. The separation principle guarantees that these can be designed and computed independently [9].

The Kalman filter is essentially a set of mathematical equations that implement a predictor-corrector type estimator that is optimal in the sense that minimizes the estimated error covariance when some presumed condition are met [10]. The Kalman filter can be thought of being a state estimator. Kalman filtering can be used as a tool to provide a reliable state estimate of the process. Another important feature of the Kalman filter is its ability to minimize the mean of the square error [11] [21].

A Loop Transfer Recovery (**LTR**) method has been used to accurately choose the tuning parameter of Kalman Filter such that **LQG** can asymptotically recover the **LQR** properties. Tuning parameters are used to improve system performance [12].

1.5. Objectives of the Thesis

- ❖ To study the dynamics of inverted pendulum system.
- ❖ To design Linear Quadratic Regulator (LQR) controller assuming all the states to be available.
- ❖ To design Kalman Filter which is an optimal observer for estimating the state vector based upon the measurement of the output and a known input for a stochastic plant.
- ❖ To design Linear Quadratic Gaussian Compensator.
- ❖ To choose the desired tuning factor value for the system by applying Loop Transfer Recovery (LTR) method.

1.6. Organization of the Thesis

The thesis contains five chapters as follows:

Chapter 1 – Introduces the classical Inverted Pendulum Control problem, its applications. It describes the Experimental Set-up. It also describes the integration between the hardware (experimental setup). Then Literature Review and Objectives of the thesis has been given.

Chapter 2- It describes the mathematical modelling of Inverted Pendulum system. It also defines the dynamics of Inverted Pendulum System. In this chapter Linear mathematical modelling has been analysed and used and after that physical constraint on Inverted Pendulum Experimental Set-up has been mentioned.

Chapter 3 –. Describes the Linear Quadratic Regulator based state feedback control law design. It describes the logic used in weight selection of the weighted matrices key to the LQR design. The chapter ends with the simulation and experimental results obtained.

Chapter 4 – This chapter starts with the definition of Linear Quadratic Gaussian (LQG) compensator design. In this chapter we have designed an optimal observer called as Kalman Filter which acts as state estimator and also considers the White Noise. Lastly Loop Transfer Recovery (LTR) has been done to analyze the system. This chapter ends with the simulation result of Kalman Filter and Simulation and Experimental Result of LQG Compensator.

Chapter 5 – Draws conclusions on the various works presented and aptly suggests the scope of future work.

Mathematical Modeling Analysis for Cart Inverted Pendulum System (CIPS)

2.1. Mathematical Modeling Analysis

A mathematical model of a dynamic system is defined as a set of equations that represents the dynamics of the system accurately. For human being, this is the best tool to describe the physical world precisely and unambiguously. The process of finding the mathematical model of a system is defined as "*mathematical modelling*". A given system can be represented by different mathematical model, provided that the model should have same input and initial conditions. Either it can be represented in "Transfer function form" or "state-space form". On one hand "transfer function form is used for SISO-LTI system, on the other hand State-Space representations with time domain analyses are used for MIMO system.

This chapter starts with the introduction of plant. Then, the complete mathematical model of the plant including Inverted Pendulum (IP) and Cart has been analyzed according to Newton's Law and then its linearized model has been presented into state space form.

2.2. Inverted Pendulum System

In this section, a full scale mathematical model for the inverted pendulum is used with a detailed explanation for each step. The motion of the inverted pendulum system consists of translational movement and rotational movement. The model of the inverted pendulum can be derived according to its movement characteristics based on the physical laws.

2.2.1. Dynamics of Inverted Pendulum System

Refer to the inverted pendulum, the system diagrammatic drawing shown in Figure 2.1. **M [kg]** is the mass of the cart; **m [kg]** is the mass of the pendulum and of the rod; **L [m]** is the length from the pivot to the center of gravity of the pendulum and the rod; **θ [rad]** is the angle between the rod and the vertical direction. **F [N]** is the force applied to the cart. **X [m]** is the displacement of the cart from the original position; **H [N]** and **V [N]** indicate the horizontal and vertical reaction forces the rod and cart. **b**= Cart friction coefficient.

I= Moment of inertia of the inverted pendulum. The vertical line on the left hand side of the diagram indicates the original position of the cart. This line is also considered as the reference position for the cart. Because the pendulum and the rod have similar motion characteristics, the analysis about the pendulum and the rod are taken as a whole.

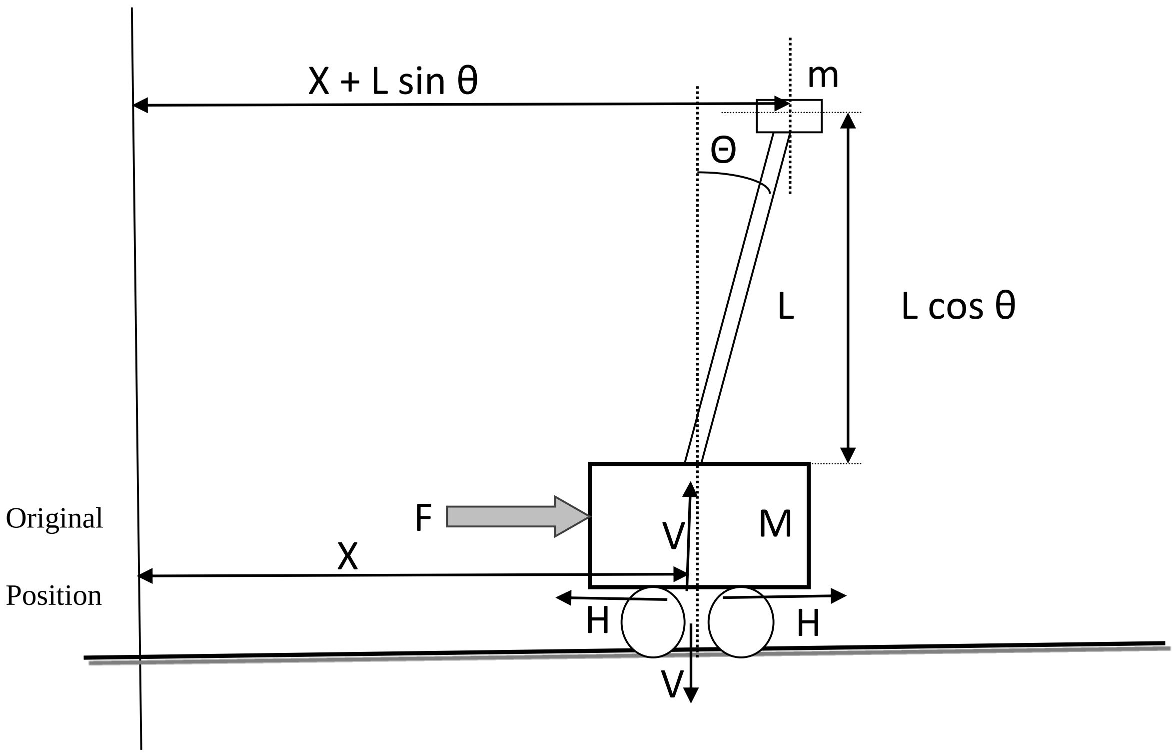


Figure 2.1.The Inverted Pendulum System Simplified Diagram

The phenomenological model (Figure 2.2) of the pendulum is nonlinear, meaning that one of the States is an argument of the nonlinear function. For such a model to present in transfer function (a form of linear plant dynamics representation used in control engineering), it has to be linearized.

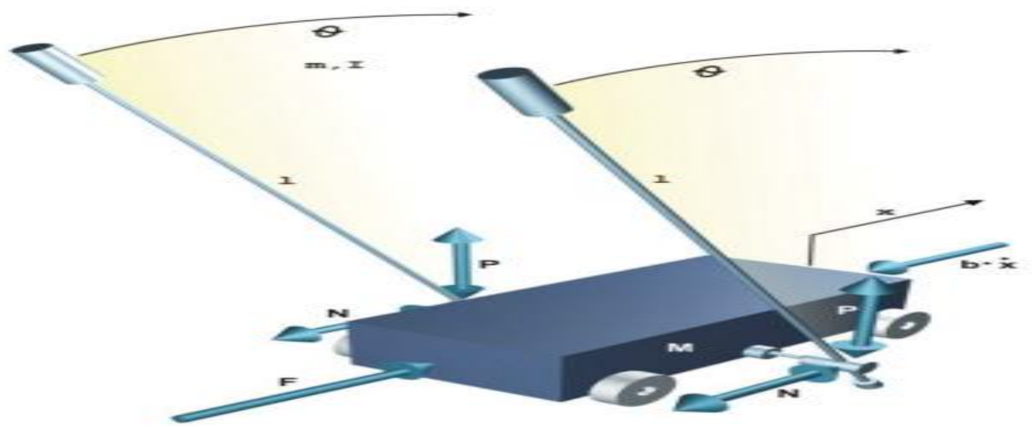


Figure 2.2. Phenomenological model of Inverted Pendulum

Motion of inverted pendulum has both translational and rotational movement. There are two approaches for modeling. The first one is Newtonian approach and the other is Lagrangian approach. Here the well-established Newtonian approach has been used.

The following is the parameter table that gives the value of the various parameters that has been adopted from the Feedback Digital Pendulum Manual [14].

Table 1.1 Inverted Pendulum System Parameters [14]

Parameters	**Values**
M- Mass of the cart in kg	**2.4kg**
m-Mass of the pendulum in kg	**0.23kg**
L-Length of pole in m	**0.36 to 0.4m**
g-Acceleration due to gravity in m/s^2	**9.81 m/s^2**

J-Moment of Inertia in kg/s²	0.099 kg/s²
b - Cart friction coefficient in Ns/m	0.05 Ns/m
b_t –Pendulum damping coefficient in N-ms/rad	0.005 N-ms/rad

Let **H** the horizontal component of reaction force and **V** be vertical component of reaction force. Let $\mathbf{X_G}$ be the horizontal component of co-ordinates of Centre of Gravity (COG) and $\mathbf{Y_G}$ be the vertical component of co-ordinates of COG.

$$\mathbf{X_G = X + L\sin\theta} \quad \textbf{(1.1)}$$

$$\mathbf{Y_G = L\cos\theta} \quad \textbf{(1.2)}$$

Let us analyze the translational motion first. Using the Newton's First law of motion we get that the net force applied on the body is equals the product of mass and its acceleration.

$$\mathbf{F = m.a} \quad \textbf{(1.3)}$$

So the horizontal reaction force **H** becomes:-

$$\mathbf{H = m.\ddot{X}_G = m.\frac{d^2}{dt^2}(X+L\sin\theta)}$$

$$\mathbf{= m\left(\ddot{X} + \ddot{\theta}L\cos\theta + \dot{\theta}^2 L(-\sin\theta)\right)} \quad \textbf{(1.4)}$$

The forced F applied on the cart equals the sum of the force due to acceleration, friction component of force that opposes the linear motion of the cart and the horizontal reaction.

$$\mathbf{F = M\ddot{X} + b\dot{X} + H} \quad \textbf{(1.5)}$$

Substituting from **(1.4)** in **(1.5)** we get:-

$$\mathbf{F = (m + M)\ddot{X} + b\dot{X} + mL\ddot{\theta}\cos\theta - mL\dot{\theta}^2\sin\theta} \quad \textbf{(1.6)}$$

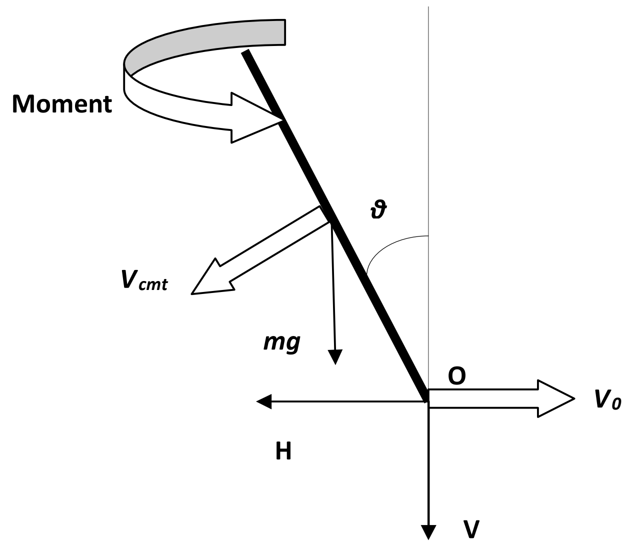


Figure 2.3. Free Body Diagram of Inverted Pendulum

Now, analyzing the rotational motion the horizontal and vertical forces in two directions one in perpendicular direction and the other in parallel direction to the rod is given by:

We get:-

$$\mathbf{V_P = V\sin\theta + mg\ \sin\theta} \tag{1.7}$$

And, $\mathbf{H_P = H\cos\theta}$ **(1.8)**

Both the forces are acting as rotational forces about the pendulum causing a rotation effect. Thus the torque equation is:-

$$\mathbf{-H\cos\theta\,.\,L + (V + mg)\sin\theta = J\ddot{\theta} + b_t\,\dot{\theta}} \tag{1.9}$$

The Vertical reaction **V** can be expressed as:-

$$\mathbf{V = m\frac{d^2}{dt^2}\,(L\cos\theta)\ =\ -m\ddot{\theta}\,L\sin\theta - m\dot{\theta}^2\,L\cos\theta} \tag{1.10}$$

Substituting from **(1.4), (1.10),** in **(1.9)** we get after rearranging:-

$$\mathbf{(J + mL^2)\ddot{\theta} =\ -mL\ddot{X}\cos\theta - b_t\,\dot{\theta} + mgL\sin\theta} \tag{1.11}$$

The equations **(1.6)** and **(1.11)** are the equations of motions for Inverted Pendulum that describe the translational and rotational motion respectively.

The state-space representation for CIPS is:

From (1.6):-

$$\ddot{\mathbf{X}} = \frac{\mathbf{F} - \mathbf{mL}\ddot{\theta}\cos\theta + \mathbf{mL}\dot{\theta}^2\sin\theta}{\mathbf{M}+\mathbf{m}}$$

Substituting for $\ddot{\mathbf{X}}$ in **(1.11):-**

$$\ddot{\theta} = \frac{\mathbf{mg}\sin\theta - \frac{\mathbf{m}}{\mathbf{M}+\mathbf{m}}\cos\theta.\mathbf{L}.\mathbf{F} + \frac{\mathbf{m}}{\mathbf{M}+\mathbf{m}}\cos\theta.\mathbf{L}.\mathbf{b}\dot{\mathbf{X}} - \frac{\mathbf{m}}{\mathbf{M}+\mathbf{m}}\cos\theta.\mathbf{L}.\mathbf{m}\,\dot{\theta}^2\,\mathbf{L}\sin\theta - \mathbf{b_t}\,\dot{\theta}}{\mathbf{J} + \mathbf{mL}^2 - \mathbf{mL}\cos\theta\,\frac{\mathbf{m}}{\mathbf{M}+\mathbf{m}}\cos\theta.\mathbf{L}} \quad \textbf{(1.12)}$$

Let the states be $\mathbf{X}, \dot{\mathbf{X}}, \theta, \dot{\theta}$:-

$$\begin{bmatrix} \mathbf{Z}_1 \\ \mathbf{Z}_2 \\ \mathbf{Z}_3 \\ \mathbf{Z}_4 \end{bmatrix} = \begin{bmatrix} \mathbf{X} \\ \dot{\mathbf{X}} \\ \theta \\ \dot{\theta} \end{bmatrix} \quad \textbf{(1.13)}$$

We have the equations:-

$$\dot{\mathbf{Z}}_1 = \mathbf{Z}_2 \quad \textbf{(1.14)}$$

$$\dot{\mathbf{Z}}_3 = \mathbf{Z}_4 \quad \textbf{(1.15)}$$

$$\dot{\mathbf{Z}}_2 = \frac{-(\mathbf{J}+\mathbf{m}\,\mathbf{L}^2)\mathbf{b}\mathbf{Z}_2 - \mathbf{m}^2\,\mathbf{L}^2\,\mathbf{g}\sin\mathbf{Z}_3\cos\mathbf{Z}_3 + \mathbf{m}\,\mathbf{L}\mathbf{b_t}\,\mathbf{Z}_4\cos\mathbf{Z}_3 + (\mathbf{J}+\mathbf{mL}^2)\mathbf{mL}\,\mathbf{Z}_4{}^2\sin\mathbf{Z}_3}{\sigma} + \frac{(\mathbf{J}+\mathbf{mL}^2)}{\sigma}\,\mathbf{F} \quad \textbf{(1.16)}$$

$$\dot{\mathbf{Z}}_4 = \frac{((\mathbf{M}+\mathbf{m})\mathbf{mgLsinZ}_3 - \mathbf{m^2\,L^2\,g\,sinZ_3cosZ_3} - (\mathbf{M}+\mathbf{m})\mathbf{b_t\,Z_4} + \mathbf{mLcosZ_3bZ_2}}{\sigma} + \frac{-\mathbf{mLcosZ_3}}{\sigma}\,\mathbf{F}$$

(1.17)

Where,

$$\boldsymbol{\sigma} = (\mathbf{J} + \mathbf{mL^2})(\mathbf{m}+\mathbf{M}) - \mathbf{m^2L^2}(\mathbf{cosZ_3})^2 \qquad \textbf{(1.18)}$$

2.2.2. Linear Mathematical Model

It is a well-known fact that more accurate the model more complex the equations will be. It is always desirable to have a simple model as it is easy to understand. So we need to strike a balance between accuracy and simplicity.

It can be seen that the equations derived above are non-linear. In order to obtain a linear model the Taylor series expansion can be used to convert the non-linear equations to linear ones and finally a given linear model will be helpful in linear control design.

Please note that the system has two equilibrium points one is the stable i.e. the pendant position and the other one is the unstable equilibrium point i.e. the inverted position. For our purpose we need to consider the second one as we require the linear model about this point. So, we assume a very small deviation θ from the vertical.

Now linearizing the model, we assume that **θ** is very small less than **5** degrees.

Therefore the following changes happen:-

$$\sin\boldsymbol{\theta} \approx \boldsymbol{\theta}, \quad \cos\boldsymbol{\theta} = \mathbf{1}, \dot{\boldsymbol{\theta}}^2 = \mathbf{0}$$

Thus the equations **(1.16), (1.17), (1.18)** changes to:-

$$\boldsymbol{\sigma}' = \mathbf{J}(\mathbf{m}+\mathbf{M}) + \mathbf{MmL^2} \qquad \textbf{(1.19)}$$

$$\dot{\mathbf{Z}}_2 = \frac{(-(\mathbf{J} + \mathbf{m}\,\mathbf{L}^2)\mathbf{b}\mathbf{Z}_2 - \mathbf{m}^2\mathbf{L}^2\mathbf{g}\mathbf{Z}_3 + \mathbf{m}\mathbf{L}\,\mathbf{b}_t\,\mathbf{Z}_4)}{\boldsymbol{\sigma}'} + \frac{(\mathbf{J} + \mathbf{m}\mathbf{L}^2)}{\boldsymbol{\sigma}'}\mathbf{F}$$

(1.20)

$$\dot{\mathbf{Z}}_4 = \frac{((\mathbf{M} + \mathbf{m})\mathbf{m}\mathbf{g}\mathbf{L}\mathbf{Z}_3 - (\mathbf{M} + \mathbf{m})\mathbf{b}_t\,\mathbf{Z}_4 + \mathbf{m}\mathbf{l}\mathbf{b}\mathbf{Z}_2}{\boldsymbol{\sigma}'} + \frac{-\mathbf{m}\mathbf{L}}{\boldsymbol{\sigma}'}\mathbf{F}$$

(1.21)

Therefore, the linearized state space model is:-

$$\begin{bmatrix} \dot{\mathbf{Z}}_1 \\ \dot{\mathbf{Z}}_2 \\ \dot{\mathbf{Z}}_3 \\ \dot{\mathbf{Z}}_4 \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{-(\mathbf{J} + \mathbf{m}\,\mathbf{L}^2)\mathbf{b}}{\boldsymbol{\sigma}'} & \frac{-\mathbf{m}^2\mathbf{L}^2\mathbf{g}}{\boldsymbol{\sigma}'} & \frac{\mathbf{m}\mathbf{L}\,\mathbf{b}_t}{\boldsymbol{\sigma}'} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \frac{\mathbf{m}\mathbf{l}\mathbf{b}}{\boldsymbol{\sigma}'} & \frac{(\mathbf{M} + \mathbf{m})\mathbf{m}\mathbf{g}\mathbf{L}}{\boldsymbol{\sigma}'} & \frac{-(\mathbf{M} + \mathbf{m})\mathbf{b}_t}{\boldsymbol{\sigma}'} \end{bmatrix} \begin{bmatrix} \mathbf{Z}_1 \\ \mathbf{Z}_2 \\ \mathbf{Z}_3 \\ \mathbf{Z}_4 \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \frac{(\mathbf{J} + \mathbf{m}\mathbf{L}^2)}{\boldsymbol{\sigma}'} \\ \mathbf{0} \\ \frac{-\mathbf{m}\mathbf{L}}{\boldsymbol{\sigma}'} \end{bmatrix} \mathbf{F}$$

(1.22)

This is the state equation and we have the output equation as:-

$$\mathbf{Y} = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{Z}_1 \\ \mathbf{Z}_2 \\ \mathbf{Z}_3 \\ \mathbf{Z}_4 \end{bmatrix} \qquad \textbf{(1.23)}$$

Hence we have obtained the state space model of inverted pendulum.

Substituting the value of parameters from Table 1.1 in (22) and neglecting cart coefficient friction, we get:-

$$\begin{bmatrix} \dot{\mathbf{Z}}_1 \\ \dot{\mathbf{Z}}_2 \\ \dot{\mathbf{Z}}_3 \\ \dot{\mathbf{Z}}_4 \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{0.0195} & \mathbf{0.2381} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & -\mathbf{0.0132} & \mathbf{6.8073} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{Z}_1 \\ \mathbf{Z}_2 \\ \mathbf{Z}_3 \\ \mathbf{Z}_4 \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{0.3895} \times \mathbf{15} \\ \mathbf{0} \\ \mathbf{0.2638} \times \mathbf{15} \end{bmatrix} \mathbf{F}$$

Here, we have considered K=15 as we know that, the D.C. Motor is used to convert control voltage, u to force, F is represented by only gain (K) =15 for simplicity [16].

The inverted pendulum is a SIMO system where the outputs are the cart position **X**, and the pendulum angle **θ**. There are two transfer functions obtained from the state space to transfer function conversion.

$$\frac{\mathbf{X(s)}}{\mathbf{F(s)}} = \frac{(\mathbf{J} + \mathbf{mL^2})\mathbf{s^2} + \mathbf{b_t\,s} - \mathbf{mgL}}{\mathbf{s((MmL^2} + (\mathbf{M} + \mathbf{m})\mathbf{J})\mathbf{s^3} + (\mathbf{bmL^2} + \mathbf{b_t}\,(\mathbf{M} + \mathbf{m}) + \mathbf{bJ})\mathbf{s^2} + (-\mathbf{mgL}(\mathbf{M} + \mathbf{m}) + \mathbf{b_t\,b})\mathbf{s} - \mathbf{mLbg}}$$

$$\frac{\boldsymbol{\theta}\mathbf{(s)}}{\mathbf{F(s)}} = \frac{\mathbf{mLs^2}}{\mathbf{s((MmL^2} + (\mathbf{M} + \mathbf{m})\mathbf{J})\mathbf{s^3} + (\mathbf{bmL^2} + \mathbf{b_t}\,(\mathbf{M} + \mathbf{m}) + \mathbf{bJ})\mathbf{s^2} + (-\mathbf{mgL}(\mathbf{M} + \mathbf{m}) + \mathbf{b_t\,b})\mathbf{s} - \mathbf{mLbg}}$$

Neglecting **b** and $\mathbf{b_t}$ we get the following simplified transfer functions:-

$$\frac{\mathbf{X(s)}}{\mathbf{F(s)}} = \frac{(\mathbf{J} + \mathbf{mL^2})\mathbf{s^2} - \mathbf{mgL}}{\mathbf{s^2}((\mathbf{MmL^2} + (\mathbf{M} + \mathbf{m})\mathbf{J})\mathbf{s^2} - \mathbf{mgL}(\mathbf{M} + \mathbf{m}))} \tag{1.24}$$

$$\frac{\boldsymbol{\theta}\mathbf{(s)}}{\mathbf{F(s)}} = \frac{\mathbf{mLs^2}}{\mathbf{s^2}((\mathbf{MmL^2} + (\mathbf{M} + \mathbf{m})\mathbf{J})\mathbf{s^2} - \mathbf{mgL}(\mathbf{M} + \mathbf{m}))} \tag{1.25}$$

Substituting from **Table 1.1,** we get the following transfer functions:-

$$\frac{\mathbf{X(s)}}{\mathbf{F(s)}} = \frac{\mathbf{0.1358s^2} - \mathbf{0.9016}}{\mathbf{s^2}(\mathbf{0.3487s^2} - \mathbf{2.3712})}$$

$$\frac{\boldsymbol{\theta}\mathbf{(s)}}{\mathbf{F(s)}} = \frac{\mathbf{0.0920s^2}}{\mathbf{s^2}(\mathbf{0.3487s^2} - \mathbf{2.3712})}$$

If we analyze the transfer functions for the poles and zeros we get to know that

For $\mathbf{X(s)/F(s)}$:-

$\mathbf{Poles = 0, 0, 6.8001, -6.8001}$

$\mathbf{Zeros = 6.6392, -6.6392}$

Here the two pole – zero pair cancels nearly leaving double pole at origin that is highly unstable.

For $\boldsymbol{\theta(s)/F(s)}$ **:-**

$\mathbf{Poles = 0, 0, 6.8001, -6.8001}$

$\mathbf{Zeros = 0, 0}$

Here also two pairs of poles and zeros cancels leaving behind an unstable pole at RHS of s plane and another at LHS of s-plane making the Transfer function highly unstable. The unit step response for the system transfer function well establishes the instability.

2.3. Physical Constraints on Inverted Pendulum Experimental Setup

There are certain Physical Constraints which are to be kept in mind while analyzing this system. In this case the limitations are:

- ❖ The distance covered by the cart from the starting point (or from the center of the Rail), i.e. x should be in the range of 0.4 meter.
- ❖ The acute angle of the pendulum w.r.t. to vertical position should be in the range of 0.2 radian.
- ❖ The applied voltage to the DC motor should remain within the range of -2.5V to + 2.5V.

Linear Quadratic Regulator (LQR) Design Applied to Cart Inverted Pendulum System-A Linear Optimal Controller

3.1. Controller Task for Cart Inverted Pendulum System

Inverted Pendulum is inherently unstable. Left without a stabilizing controller, it will not be able to remain in an upright position when disturbed. The controller task will be to change the D.C. voltage depending on the two variables Pendulum Position (angle) and the Cart Position on the rail, in such a way that the desired control task is fulfilled (stabilizing in an upright position, swinging or crane control).

3.1.1. Why Quadratic Optimal Controller is Preferred over Other Available Controllers?

This project is based on a linear model of Inverted pendulum. Here, a linearized plant is considered and linear controller are designed based on quadratic optimal control method. Although another linear controller design method is available like Pole-Placement method. But there are various advantages of optimal regulator method over pole placement method. They are:

- ❖ Optimal controller method provides a systematic way of computing the state feedback control gain matrix (Ogata, 2002:827).
- ❖ In pole-placement method the closed loop pole location must be determined, but the researcher may not really know where they are located. The optimal control method ignores finding the desired pole location.
- ❖ For the same system, there is not unique control law based on the pole-placement method. The designer may not know how to choose the best one. The control law of the optimal control method always optimizes performance of the system in the accurate sense and the above all drawbacks are avoided.

Now-a-days various other good controller design method are available for the control engineers. Some of these methods are 'The Proportional – Integral –Derivative (PID) and Proportional-Derivative (PD) controller [17] and [18] and Fuzzy control [19] to mention a few. But one of the obstacles by using the PID and PD controller are that they alone cannot effectively control all of the pendulum state variables since they are of lower order than the pendulum itself. They are usually replaced by a full order. So, a linear state feedback controller based on the linearized Inverted Pendulum model can instead be used and may also be extended with a disturbance observer (Kalman Filter), to improve the disturbance rejection performance, which can be analyzed in chapter-4.

3.2. Linear Quadratic Regulator (LQR)

Optimal control provides an alternative design strategy by which all the control design parameters can be determined even for multi-input, multi-output (MIMO) system. It allows us to directly formulate the performance objectives of a control system. Moreover, it produces the best possible control system for a given set of performance objectives.

The LQR is one of the most widely used and simplest static state feedback method, primarily as the LQR based pole placement helps us to translate the performance constraints into various weights in the performance index.

3.2.1. How LQR Works?

LQR is that optimal controller where the objective function is a time integral of the sum of transient energy and control energy expressed as function of time and thus here we try to minimize that particular objective function by suitably selecting the performance and control cost weighting matrices, Q and R and solving the Riccatti equation subjected to terminal condition in order to determine the optimal regulator gain, K.

A state feedback can be generalized for an LTI system is given below:

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{Ax} + \mathbf{Bu} \\ \mathbf{y} &= \mathbf{Cx} \end{aligned} \tag{3.1}$$

Here while working with LQR we assume that all the n states are available for feedback and the states are completely controllable then there is a feedback gain matrix K, such that the state feedback control input is given by

$$\mathbf{u} = -\mathbf{K}\left(\mathbf{x} - \mathbf{x_d}\right) \tag{3.2}$$

Let $\mathbf{x_d}$ be the desired states vector.so, closed loop system dynamics using (3.2) in (3.1) becomes

$$\dot{x} = (A - BK)x + BKx_d \tag{3.3}$$

Here, value of optimal regulator gain K depends on the desired pole locations where one wants to place the poles to achieve the desire control performance. In LQR the control is subjected to a Performance Index (PI) or Cost Functional (CF) given by

$$J = \frac{1}{2}\left[z(t_f) - y(t_f)\right]^T F(t_f)\left[z(t_f) - y(t_f)\right] + \frac{1}{2}\int_{t_0}^{t_f}\left\{[z-y]^T Q[z-y] + u^T Ru\right\}dt \tag{3.4}$$

Here z is the m dimensional reference vector and u is an r dimensional input vector. If all the four states in case of Inverted Pendulum Controller design are available in the output for feedback then m equals n. In (3.4), the matrix Q is known as the state weighted matrix that penalizes certain states, R is the control cost weighted matrix that penalizes control inputs, F is known as the terminal cost weighted matrix. The following conditions (sufficient but not necessary) may be satisfied for the LQR implementation or for the existence of solution of Algebraic Riccatti Equation:-

- The plant with coefficient matrices A, B must be controllable.
- All the weighted matrices Q and R are square symmetric in nature.
- The state weighted matrix Q must be symmetric and positive semi-definite as to keep the error squared positive. Due to quadratic nature of PI, more attention is being paid for large errors than small ones. Usually it is chosen as a diagonal matrix.
- The control weighted matrix R is always symmetric positive definite i.e. all the eigen values of R must be positive real numbers as the cost to pay for control is always positive. One has to pay more cost for more control.
- The terminal cost weighted $F(t_f)$ is to ensure that the error e(t) reaches a small value in a finite time t_f .So the matrix should always be positive semi-definite.

Here, an Infinite Time LQR problem has been used where the final end cost $F(t_f)$ is zero at $t_f = \infty$.

For infinite final time, the Quadratic objective function can be expressed as follows:

$$J_\infty = \int_{t_0}^{\infty} \frac{1}{2}\left\{x^T Qx + u^T Ru\right\}dt \tag{3.5}$$

On applying Pontryagin's Maximum Principle on the open loop system an optimal solution for the closed loop system we have obtained the following equations:-

$$\begin{aligned} &\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu},\ \mathbf{x}(\mathbf{t_0}) = \mathbf{x_0} \\ &\dot{\boldsymbol{\lambda}} = -\mathbf{Qx} - \mathbf{A^T}\boldsymbol{\lambda},\ \lambda(\mathbf{t_f}) = \mathbf{0} \\ &\mathbf{Ru} + \mathbf{B^T}\boldsymbol{\lambda} = \mathbf{0} \end{aligned} \tag{3.6}$$

Since all the equations in (3.6) are linear these can be connected by

$$\boldsymbol{\lambda} = \mathbf{Mx} \tag{3.7}$$

Here, M is the solution to Algebraic Riccati Equation. However, solution to ARE may not always exists. By substituting for $\dot{\boldsymbol{\lambda}}$ from (3.6) and then substituting for $\dot{\mathbf{x}}$ from (3.6) and using (3.7) by substituting for u from (3.6) we get

$$\mathbf{MAx} + \mathbf{A^T Mx} + \mathbf{Qx} - \mathbf{MBR^{-1}B^T Mx} + \dot{\mathbf{M}} = \mathbf{0} \tag{3.8}$$

This is called Matrix Riccati Equation.

Now, the solution at steady state is given by Algebraic Riccati Equation (ARE) as given below:-

$$\mathbf{MA} + \mathbf{A^T M} + \mathbf{Q} - \mathbf{MBR^{-1}B^T M} = \mathbf{0} \tag{3.9}$$

The optimal feedback gain matrix is obtained from $\mathbf{Ru} + \mathbf{B^T}\boldsymbol{\lambda} = \mathbf{0}$ as given below:-

$$\begin{aligned} \mathbf{u} &= -\mathbf{R^{-1}B^T Mx} \\ &= -\mathbf{Kx} \end{aligned} \tag{3.10}$$

Where, K is optimal feedback gain

3.2.2. Properties of LQR

LQR has many desirable properties. They are following:

- For all frequencies, the Nyquist plot of the open-loop transfer function of an LQR-based design always stays outside a unit circle centered at (-1,0).
- LQR solution, in SISO case, has at least 60^0 phase margin, infinite gain margin and a gain reduction tolerance of -6 dB.
- LQR solution is its high-frequency Roll-off rate.

3.3. LQR Controller Design

The design strategy used here is by LQR (linear quadratic regulator) method. An LQR controller is designed considering both pendulum's angle and cart's position. The four states are assumed to be available. These four states represent the position, velocity of the cart, angle and angular velocity of the pendulum. The output y contains both the position of the cart and the angle of the pendulum. A controller is to be designed such that, when the pendulum is displaced, it eventually returns to zero angle (i.e. the vertical) and the cart should be moved to a new desired position according to the controller.

The next step in designing such a control is to determine the feedback gains, K.

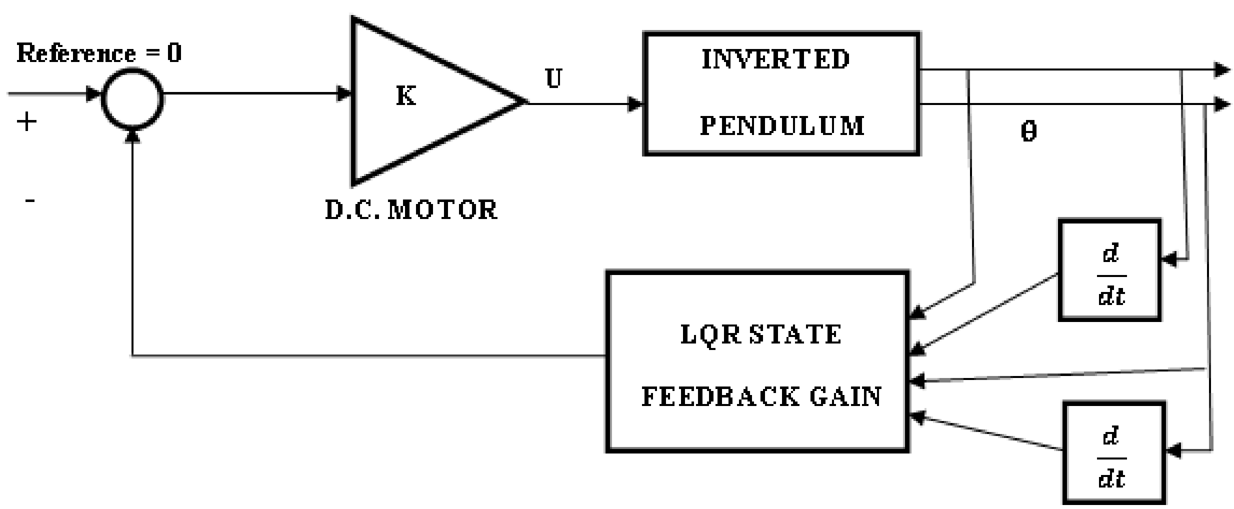


Figure.3.1. Block Diagram of LQR Controller

The K matrix can be produced by choosing a suitable value of Q and R using MATLAB command. Q and R matrix is adjusted by iterative method to obtain the desired response by satisfying certain conditions. These conditions are presented here in the form of an Algorithm, these are:

- Using LQR function, two parameters i.e. R and Q can be chosen, which will balance the relative importance of the input the element at row 1, column 1 in Q matrix weights to the position of the cart. Similarly the element at row 2, column 2 weights to the velocity of the cart, element at row 3 column 3 weights to the pendulum angle, element at row 4 column 4 weights to the angular velocity of the pendulum. R gives weight to the input voltage.
- Since there is constraint on position of the cart that it has to be in between -0.4 to 0.4 m so this factor is of utmost important to us so we will give weightage to it more.so here, $\mathbf{q_1 >> q_2, q_3, q_4}$.
- To fix the pendulum in the upright position, the cart has to move rapidly i.e. cart velocity should change faster than angular velocity to keep the pendulum hold.so here, $\mathbf{q_2 >> q_4}$.
- To keep the control voltage minimum we should choose **R>>1.**

3.4. Limitations of LQR

Limitation of LQR design are [20]:

- Full state feedback requires all the states to be available. This limits the use of LQR in flexible structures as such systems would infinite number of sensors for complete state feedback.
- The LQR is an optimal control problem subjected to certain constraints so the resultant controller usually do not ensure disturbance rejection as it indirectly minimizes the sensitivity function, reduction in overshoot during tracking, stability margins on the output side etc.
- Optimality does not ensure performance always.
- LQR design is entirely an iterative process that as the LQR doesn’t ensure standard control system specifications, even though it provides optimal and stabilizing controllers. Hence, several trial and error attempts is required to ensure satisfactory control design.

3.5. Results and Discussions

Both, the simulation and experiment are conducted using a second order derivative filter F of cutoff frequency 100 rad/s and damping ratio 0.35. By trial and error , **Q** can be chosen as a diagonal **4X4** matrix and **R** is a **scalar** as only a single control input exists were determined to be as the following:

$$\mathbf{Q} = \begin{bmatrix} \mathbf{1250000} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{75000} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{75000} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{2500} \end{bmatrix}, \mathbf{R} = \mathbf{10^{2.6}}$$

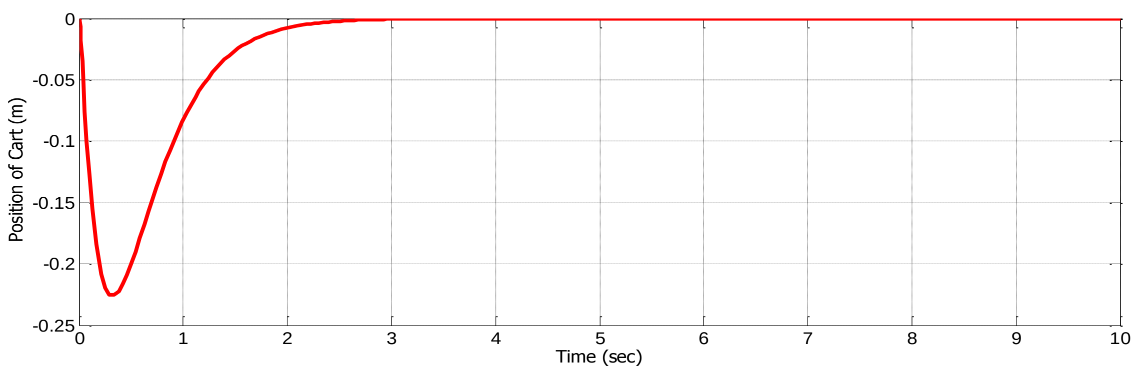
And, a MATLAB m-file was written to calculate the LQR gain using the command **lqr(A,B,Q,R)**. The obtained LQR gains are:

$$\mathbf{K} = \begin{bmatrix} \mathbf{-56.0344} & \mathbf{-57.4338} & \mathbf{277.4261} & \mathbf{107.4952} \end{bmatrix}$$

The simulation and experimental results are shown below:

3.5.1. Simulation Results of LQR

Fig. 3.2 shows the time response of a system in simulation. The simulation result for initial condition [0 0 0.1 0] for the LQR scheme for the cart position, linear velocity of the Cart, angle of Pendulum, angular velocity and control voltage .Here it can be seen that displacement reaches its final value in less than 3 seconds and the system has better stability. The speed of reaching the final value depends on choice of Q matrix. Choosing high value of Q means having faster response for any input signal and having better stability. To keep the control voltage minimum we should choose **R>>1.** Here, we can analyze from figure that Inverted Pendulum constraints have been satisfied as maximum position of Cart doesn't go beyond **0.4 m** and control voltage is in the range of **-2.5v to +2.5v.**



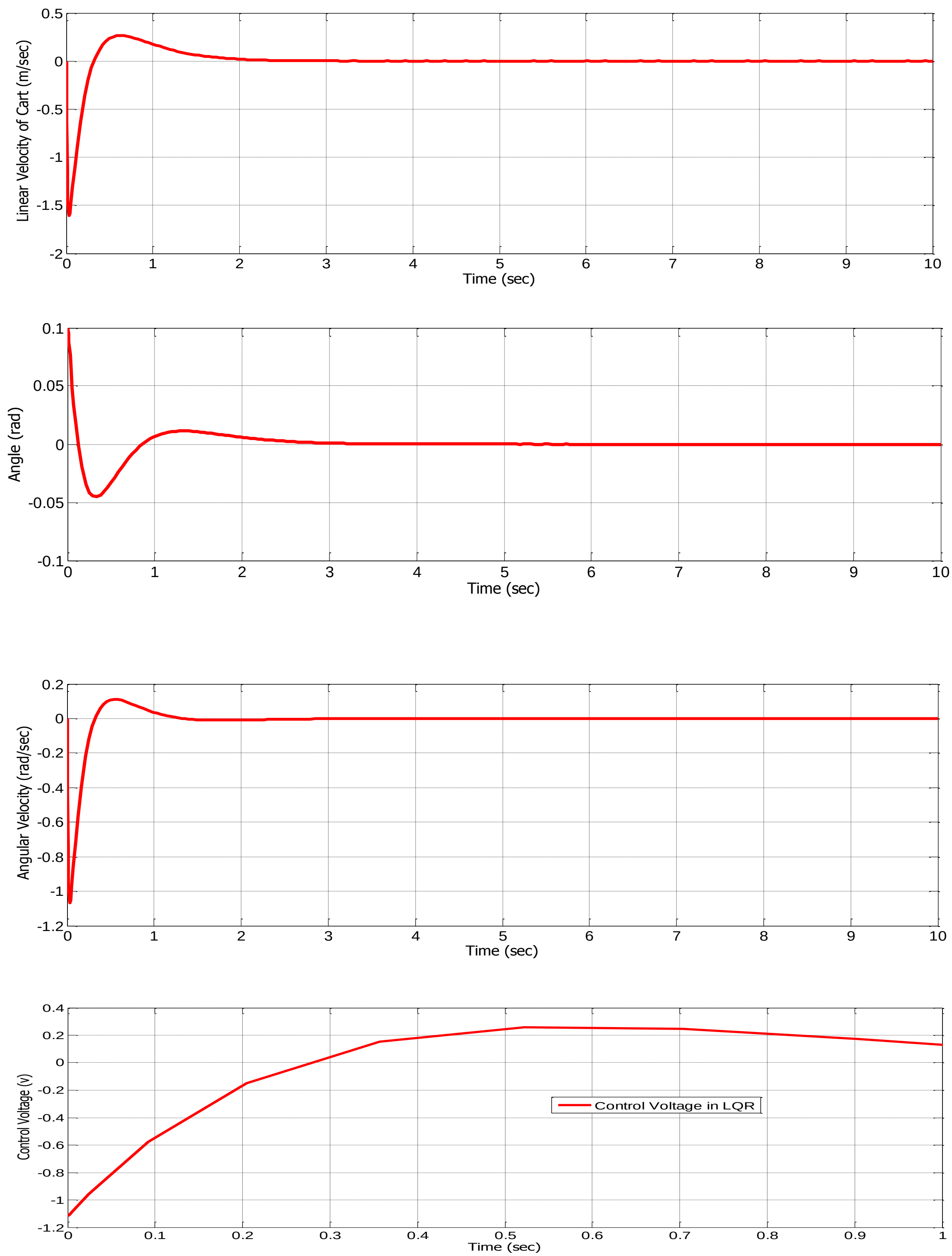


Figure 3.2. Simulation Results of all the four states & Control Voltage using LQR

3.5.2. Experimental Results of LQR

Figure 3.3. Shows the Experimental Result for initial condition [0 0 0.1 0] of Cart Position and Angle of LQR. Here, we can observe that Cart is oscillating about the mean position in order to make the pendulum hold in its upright position i.e. maintaining an angle of 0 radian. The figure shows some undesired oscillation in real time. This may be due to due to Non-linear friction behaviour that causes friction memory like behaviour or low frequency noise.

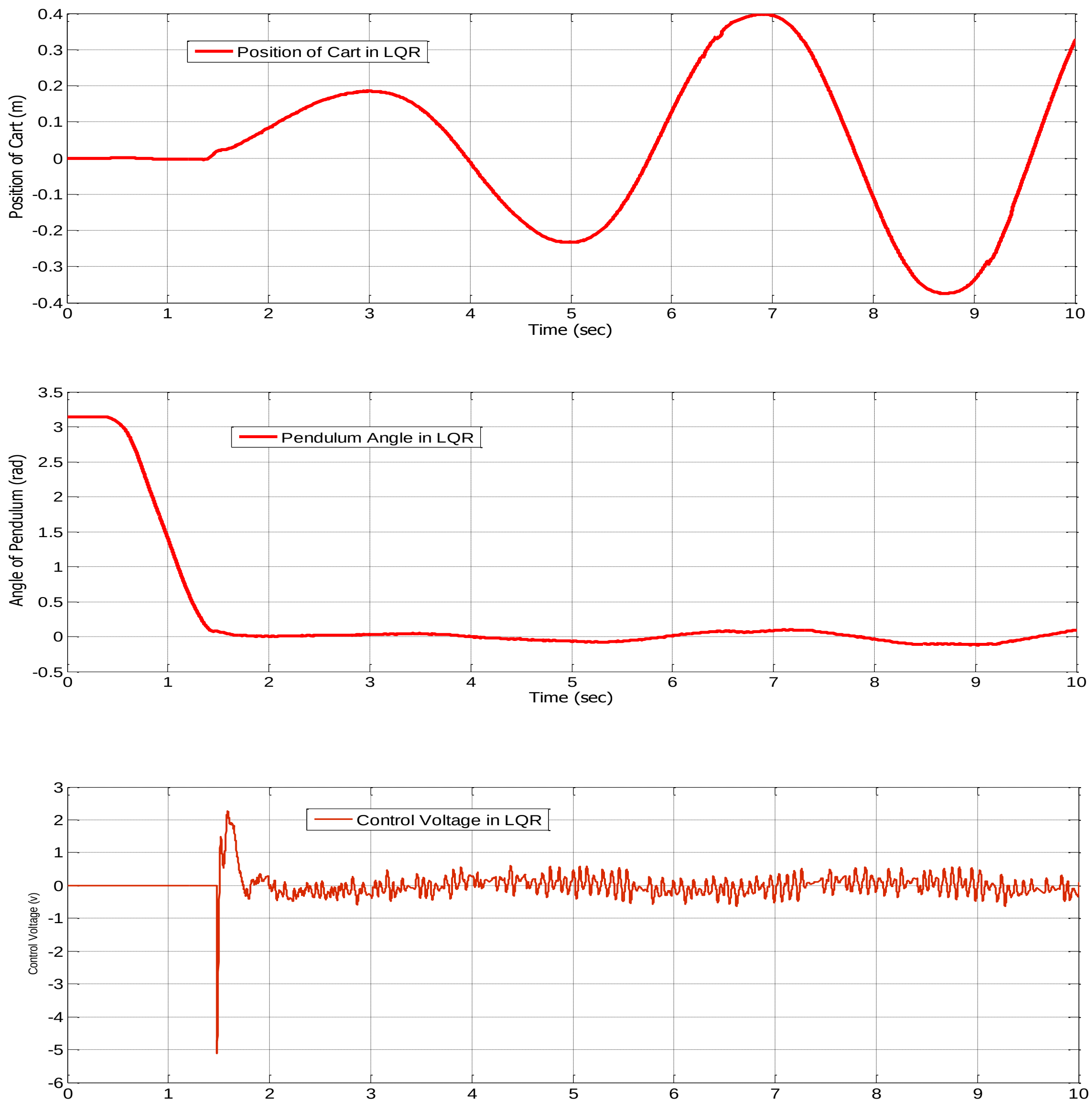


Figure 3.3.Experimental Results of Available States and control Voltage at Output of LQR

Linear Quadratic Gaussian (LQG) Compensator Design Applied to Cart Inverted Pendulum System

4.1. Introduction

LQG is a type of compensator. A compensator is a combination of separately designed regulator and an observer using pole-placement. More precisely describing a LQG, it is a combination of Linear Quadratic Regulator (LQR) designed in last chapter (chapter-4) with that of Kalman Filter. Here LQR is a linear regulator that minimized a Quadratic Objective Function, which includes transient, terminal, and control penalties. Kalman Filter is an optimal observer for multi output plant in the presence of process and measurement noise, modeled as white noise. The task of optimal observer such as Kalman Filter is to estimates the states. Since the optimal compensator is based upon a linear plant, a quadratic objective function, and an assumption of white noise that has a normal , or Gaussian , probability distribution , the optimal compensator is popularly called the Linear, Quadratic, Gaussian (or LQG) compensator [2].

4.1.1. Features of LQG

Following are the features of LQG Compensator [12]:

- LQG has better Noise separation properties.
- The controller of Linear Quadratic Gaussian (LQG) is a traditional control method for stochastic system and.
- The controller LQG is an effectively technique to solution of the optimal control problem.
- LQG will exhibit the separation property.
- LQG solution results in an asymptotically stable closed-loop system. In addition it minimizes the average of the LQR cost function (i.e., the weighted variance of the state and input).

4.2. LQG Compensator Design

The optimal compensator design process is mentioned below in three steps:

Step 1

Design an optimal regulator for a linear plant assuming full-state feedback (i.e. assuming all the state variables are available for measurement) and a quadratic objective function, **J**. The regulator is designed to generate a control input, **u,** based upon the measured state-vector, x**.**

Step 2

Design a Kalman Filter for the plant assuming a known control input, u, a measured output, y, and white noises, V and Z, with known power spectral densities. The Kalman Filter is designed to provide an optimal estimate of the state vector, $\mathbf{x}_0$.

Step 3

Combine the separately designed optimal regulator and Kalman Filter into an optimal Compensator, which generates the input vector , u , based upon the estimated state vector , $\mathbf{x}_0$, rather than the actual state-vector , x , and the measured output vector ,y .

Here, Step 1 has already been explained in a detailed manner in chapter 3. Here in this chapter we will begin with step 2 of LQG design process i.e. Kalman Filter designing and then we will combine separately designed **LQR** as per in Step 1 with the **Kalman Filter** as per in Step 2 to form the Step 3 of designing **LQG**.

4.2.1 The Kalman Filter and its Design Analysis

An Overview

Since its introduction in the early 1960s, the Kalman Filter has being widely used in the control engineering community. It can be thought of being a tool to provide a reliable state estimate of the process and also it has ability to minimize the mean of the square error and provide a solution for the least square method. So, we can use Kalman Filter on a control system i.e. exposed to noisy environment. The LQR solution is basically a state-feedback type of controller –i.e., it requires that all states be available for feedback. This was urgued in the previous chapter that this is usually an unreasonable assumption and some form of state estimation is necessary. Hence, Kalman Filter does the same task of estimation of states on the basis of observed output and control input. Kalman Filter is basically an observer (optimal observer). Kalman Filter is also known as Linear Quadratic Estimator (LQE).

Design Analysis of Kalman Filter

Before using a Kalman Filter, the behavior of the system that we are measuring must be described by a Linear System. A Linear Time Variant system is described by the two equation System equation and Output equation.

$$\dot{\mathbf{x}}(\mathbf{t}) = \mathbf{A}(\mathbf{t})\mathbf{x}(\mathbf{t}) + \mathbf{B}(\mathbf{t})\mathbf{u}(\mathbf{t}) + \mathbf{F}(\mathbf{t})\mathbf{v}(\mathbf{t}) \tag{4.1}$$

$$\mathbf{y}(\mathbf{t}) = \mathbf{C}(\mathbf{t})\mathbf{x}(\mathbf{t}) + \mathbf{D}(\mathbf{t})\mathbf{u}(\mathbf{t}) + \mathbf{z}(\mathbf{t}) \tag{4.2}$$

Here, x is the state of the system and y is the measured o/p and u is the known input of the system.

A, B, C are matrices that give value to their related system.

$\mathbf{v}(\mathbf{t})$ is Process Noise Vector given by $\boldsymbol{\rho} * \mathbf{f}^{\mathbf{T}} * \mathbf{f}$ which arises due to modeling errors such as neglecting non-linear or higher frequency dynamics

And, $\mathbf{z}(\mathbf{t})$ is Measurement Noise Vector given by $\boldsymbol{\sigma} * \mathbf{C}^{\mathbf{T}} * \mathbf{C}$

Both Noises are assumed to be White Noise. $\mathbf{v}(\mathbf{t})$ & $\mathbf{z}(\mathbf{t})$ can be expressed as follows:

$$\mathbf{R}_{\mathbf{V}}(\mathbf{t},\boldsymbol{\tau}) = \mathbf{V}(\mathbf{t})\boldsymbol{\delta}(\mathbf{t}-\boldsymbol{\tau}) \tag{4.3}$$

$$\mathbf{R}_{\mathbf{Z}}(\mathbf{t},\boldsymbol{\tau}) = \mathbf{Z}(\mathbf{t})\boldsymbol{\delta}(\mathbf{t}-\boldsymbol{\tau}) \tag{4.4}$$

Where, $\mathbf{V}(\mathbf{t})$ & $\mathbf{Z}(\mathbf{t})$ are the time varying power spectral density matrices of $\mathbf{v}(\mathbf{t})$ & $\mathbf{z}(\mathbf{t})$.

$\mathbf{R}_{\mathbf{V}}(\mathbf{t},\boldsymbol{\tau})$ & $\mathbf{R}_{\mathbf{Z}}(\mathbf{t},\boldsymbol{\tau})$ are infinite covariance matrices respectively, which can be regarded as a characteristics of White Noise-stationary or non-stationary.

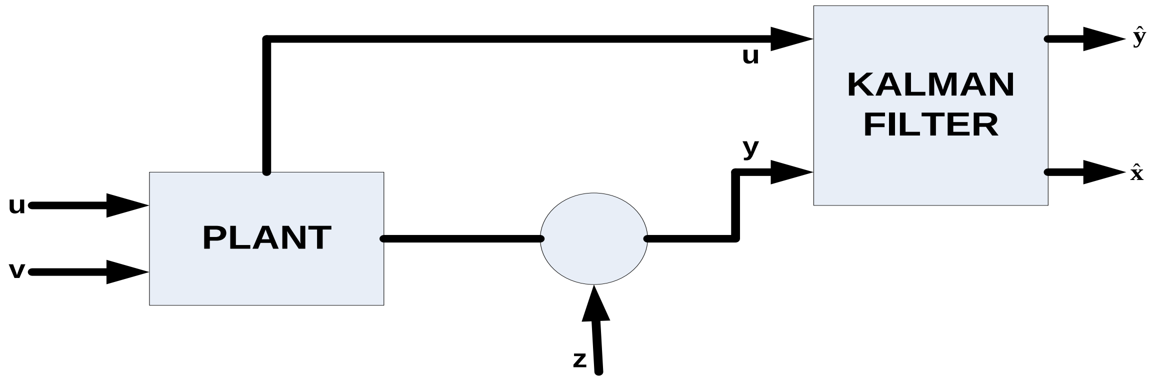


Figure 4.1. Block Diagram of Kalman Filter

The state space solution of the problem was first provided by R. F. Kalman and R. S. Bucy. The Kalman Filter minimizes a statistical measure of the estimation error:

$$\mathbf{e_0(t) = x(t) - x_0(t)} \tag{4.5}$$

Where, $\mathbf{x_0(t)}$ is the estimated state vector.

The state equation of the Kalman Filter is that of a time varying observer and can be written as follow:

$$\mathbf{\dot{x}_0(t) = A(t)x_0(t) + B(t)u(t) + L(t)\left[y(t) - C(t)x_0(t) - D(t)u(t)\right]} \tag{4.6}$$

Where, $\mathbf{L(t)}$ is the gain matrix of the Kalman Filter (also called the optimal observer gain matrix). Basically, the Kalman Filter minimizes the covariance of the estimation error,

$$\mathbf{R_e(t,t) = E\left[e_0(t)e_0^T(t)\right]} \tag{4.7}$$

The optimal Kalman Filter gain can be computed (assuming noise signals are uncorrelated) as:

$$\mathbf{L^0(t) = R_e^0(t,t)C^T(t)z^{-1}(t)} \tag{4.8}$$

Where, $\mathbf{R_e^0(t,t)}$ is the optimal covariance matrix satisfying the following Matrix Riccati equation.

$$\mathbf{dR_e(t,t)/dt = A(t)R_e^0(t,t) + R_e^0(t,t)A^T(t) - R_e^0(t,t)C^T(t)z^{-1}(t)C(t)R_e^0(t,t) + F(t)V(t)F^T(t)} \tag{4.9}$$

But, here we are interested in steady state Kalman Filter, i.e. the Kalman Filter for which the covariance matrix converges to constant in the limit $t \to \infty$. In such a case, the following Algebraic Riccati Eqn. results for the optimal covariance matrix, $\mathbf{R_e^0}$

$$\mathbf{0 = AR_e^0 + R_e^0A^T - R_e^0C^Tz^{-1}CR_e^0 + FVF^T} \quad \mathbf{(4.10)}$$

Here, the system with state-dynamic matrix, A, and observation matrix, C, is detectable, and the system with state-dynamics matrix, A, and control coefficient matrix, $\mathbf{B = FV^{1/2}}$ is Stabilizable. This condition will be met if the system with state dynamics matrix, A and observation matrix, C is observable, V is a positive semi-definite matrix and z is a positive definite matrix.

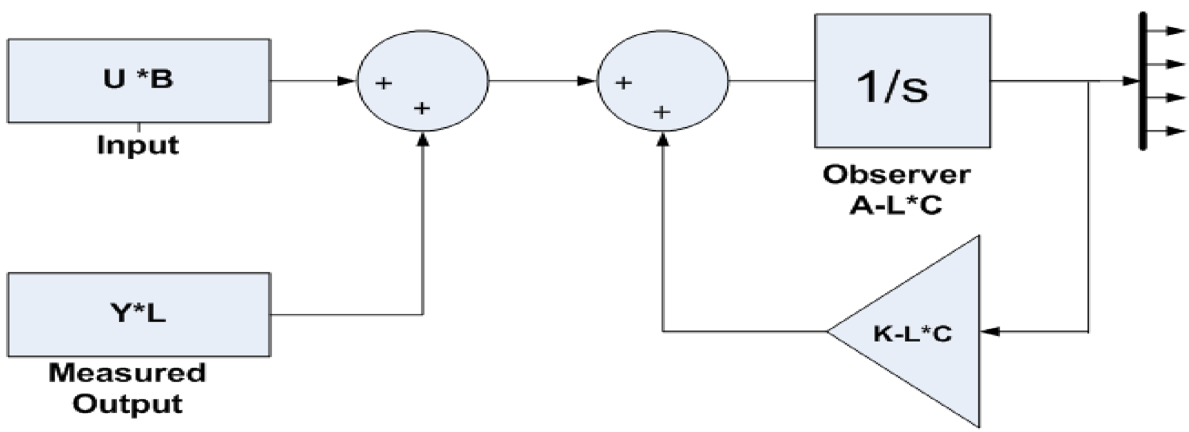


Figure 4.2.Simulation Block Diagram of Kalman Filter

4.2.2. LQG compensation- A Combination of LQR and Kalman Filter

Since the optimal regulator and Kalman Filter are designed separately, they can be selected to have desirable properties that are independent of one another .The closed loop eigen values consist of the regulator eigen values and the Kalman Filter eigen values. The block diagram of LQG compensator with the plant is shown in figure 4.3. Now the plant contains the process and measurement noise. The closed loop system's performance can be obtained as desired by suitably selecting the optimal regulator's weighting matrices, **Q** and **R**, and the Kalman Filter's spectral noise densities , **v** , **z** , ψ . Hence, the matrices **Q, R, v**, **z**, ψ are the design parameters for the closed-loop system with an optimal compensator.

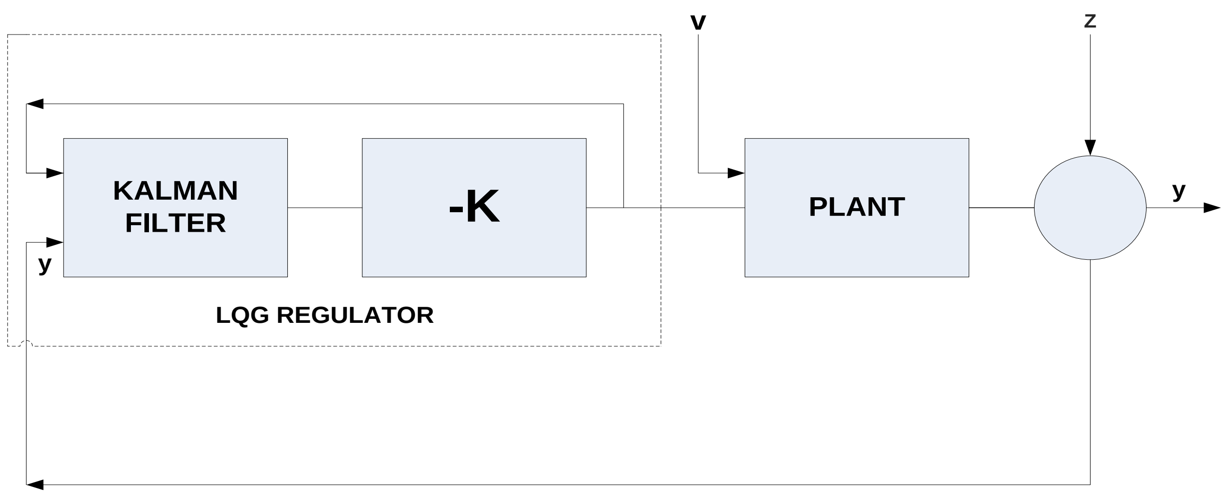


Figure 4.3. Block diagram of LQG compensator

$$\mathbf{x_0(t) = [A - BK - LC - LDK]x_0(t) + Ly(t)} \tag{4.11}$$

$$\mathbf{u(t) = -Kx_0(t)} \tag{4.12}$$

Where K and L are Optimal Regulator gain and Kalman Filter gain matrices [8].

Here,

$\mathbf{K = \begin{bmatrix} -56.0344 & -57.4338 & 277.4261 & 107.4952 \end{bmatrix}}$ And,

$$\mathbf{L = \begin{bmatrix} 10.9354 & 0.0112 \\ 9.7915 & 0.2529 \\ 0.0112 & 11.7392 \\ 0.0021 & 18.9041 \end{bmatrix}}$$

4.3. Robust Multivariable LQG Control: Loop Transfer Recovery

It was discussed earlier that the LQR solution has excellent stability margins (infinite gain margin and 60^0 phase margin). We know that LQR is usually, but not always considered impractical because it requires that all the states be available for feedback. Doyle and Stein showed that under certain conditions, the LQG can asymptotically recover the LQR properties. One of the problems with LQG is that it requires statistical information of the noise process. Mathematical arguments and simulation had shown that the LQG design parameters (Q, R and ρ) have a strong influence of the performance of the system.so they should be used as tuning parameters for improving the system performance [9].

The following are the procedure for design is suggested by the foregoing conditions. Choose the LQR parameters such that the LQR loop transfer function has desirable time and/or frequency domain properties. First of all design a Kalman Filter with parameters specified. Then, increase the tuning parameter ρ until the resulting loop transfer function is as close as possible to the TFL. When the loop transfer function of LQG approaches that of LQR, it will asymptotically recover its properties. Increase the tuning parameters, ρ the closer the LQG system comes to LQR performance. It should be noted that the value of ρ should not be increased indefinitely, because this may lead to unreasonably large value of filter gain L. whereas on the other hand smaller values of Q will tend to trade off lower stability margins with higher roll-off rates at high frequencies.

We will now vary the tuning parameters ρ over the range (1, 10, 100, 1000, 10000, 100000, 1000000) and we will then observe its effect on closed loop poles of the filter (A-L*C), Kalman Filter Gain (L), Gain Margin (G.M.), Phase Margin (PM), Phase Cross Over Frequency (w_g) and Gain Cross Over Frequency (w_p) and then we will analyse the reason of taking the specific tuning parameter, ρ

From Table 4.1. We can observe that as ρ increases, the Kalman Filter Gain (L), its Eigen values, the stability margins (G.M. and P.M.), Gain Crossover Frequency and Phase Crossover Frequency also increases.

Table 4.1. Result of LTR Design

Tuning Parameter (ρ)	Closed Loop Poles (A-L*C)	Kalman Filter Gain (L)	Gain Margin (dB)	Phase Margin (degree)	Gain Crossover freq. (w_p) (rad/sec)	Phase Cross Over Frequency (w_g)(rad/sec)
1000	[-0.42 ± 0.36i; -2.46; -2.76]	[0.84 0.13; 0.31 0.45; 0.13 5.22; 0.34 13.61]	4.01	44.84	24.55	79.97
10000	[-0.86 ± 0.50i; -2.20; -3.12]	[1.71 0.09; 0.97 0.43; 0.09 5.32; 0.25 13.60]	4.75	50.77	35.19	115.01
100000	[-4.57; -2.97; -1.66; -1.64]	[4.02 0.05; 3.09 0.37; 0.05 6.21; 0.14 14.31]	5.20	57.23	47.59	173.99
1000000	[-9.94; -10.59; 1.00, -1.14]	[10.93 0.01; 9.79 0.25; 0.01 11.73; 0.00 8.90]	6.47	64.07	59.80	274.79

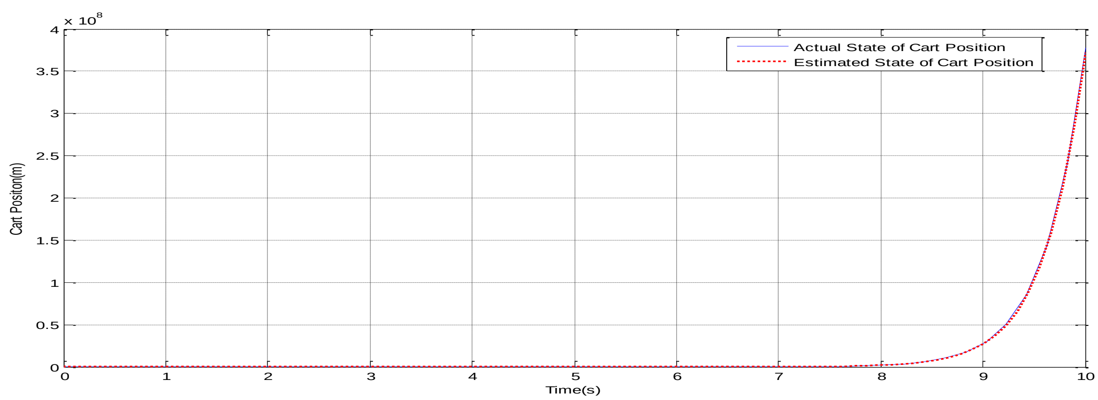
But here increasing the margin cost us in terms of higher values of filter gain, which is a drawback but there are other advantages also that on increasing the tuning parameters the closed loop poles goes far away from origin.so the dominance of closed loop poles of filter gets reduced, which is a major factor in designing the LQG compensator. But on the other hand increased value of filter gain (L), higher gain cross over frequency (w_p) will make the system more sensitive to noise and uncertainties at higher frequencies. But we can observe on the other hand that gain margin and phase margin are increasing as the tuning factor gets increases down the table and so relative stability will also increase. Also, the speed of response of system will be increasing with increasing tuning factor.so, by LTR, a set of possible tuning parameters representing the state and process noise covariances can be selected, depending on the trade-off between noise suppression and system robustness. In this project $\boldsymbol{\rho}$ =1000000 is a reasonable compromise.

4.4. Results and Discussions

4.4.1. Estimated Graphical Result of States by Kalman Filter

Fig.4.4. shows only the two estimated states i.e. Position of cart, Angle of pendulum obtained by Kalman Filter. Here, we are estimating all the four states because we have considered the Full-State observer. And, then all the estimated states have been compared by the actual states of the linearized model of plant. As we can observe from figure that all the four states are exactly overlapping with the actual states of linearized plant at $\boldsymbol{\rho}$ =1000000.since, our plant is unstable so the estimated states obtained are also unstable.

So, here $\mathbf{L} = \begin{bmatrix} \mathbf{10.9354} & \mathbf{0.0112} \\ \mathbf{9.7915} & \mathbf{0.2529} \\ \mathbf{0.0112} & \mathbf{11.7392} \\ \mathbf{0.0021} & \mathbf{18.9041} \end{bmatrix}$



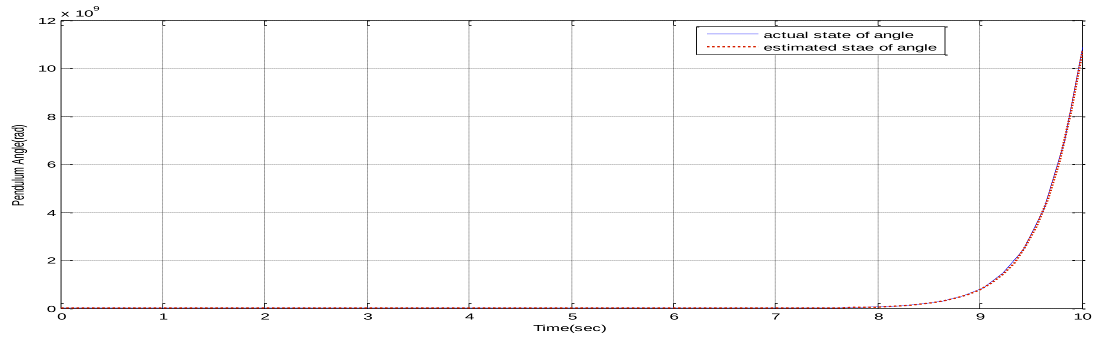


Figure 4.4. Comparison of Estimated States of Kalman Filter with the Actual States

4.4.2. Simulation Results of LQG in Comparison to LQR

Fig. 4.5. Shows the time response of a system. The simulation result for initial condition [0 0 0.1 0] for the LQR and LQG scheme for the cart position and angle. Here, it can be seen that displacement reaches its final value in less than 3 seconds and the system has better stability. The speed of reaching the final value depends on choice of Q matrix. Choosing high value of Q means having faster response for any input signal and having better stabilit

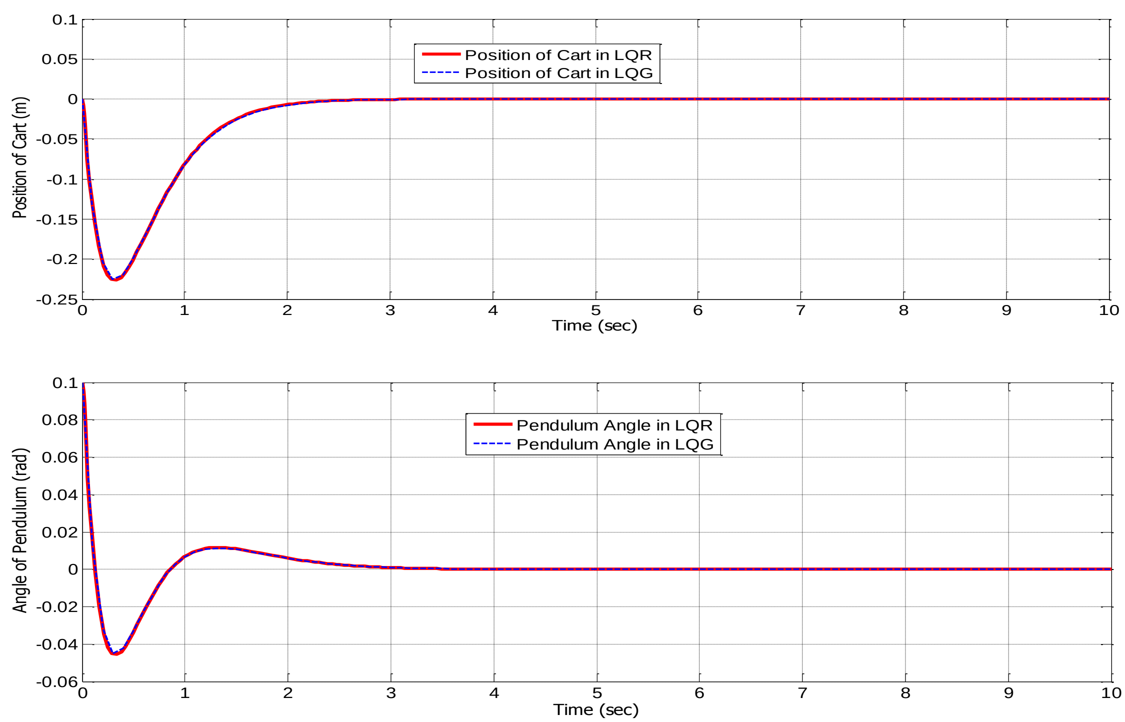


Figure 4.5.Time response of the Inverted Pendulum system for position and the angle of the Cart

4.4.3. Experimental Results of LQG in Comparison to LQR

Fig 4.6 .shows the Experimental results for initial condition [0 0 0.1 0] of Cart Position, Angle of LQR/LQG and Control Voltage. The figure shows some undesired oscillation in real time. This may be due to Non-linear friction behaviour that causes friction memory like behaviour or low frequency noise.

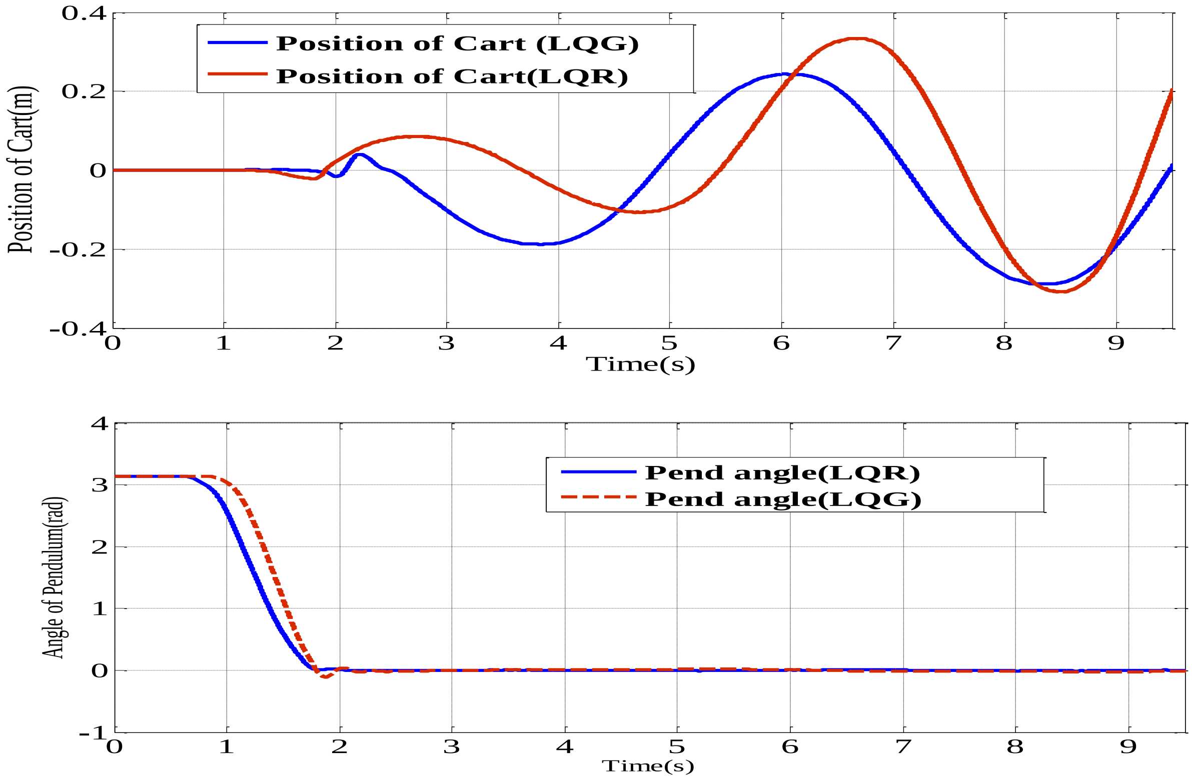


Figure 4.6. Experimental Time response of the inverted pendulum system for Position and Angle of Cart.

CONCLUSIONS AND FUTURE WORK

5.1. Conclusions

The thesis presents a number of control approaches such as LQR and LQG for CIPS. These design methods have been successful in meeting the stabilization goals of the CIPS, simultaneously satisfying the physical constraints in track limit and control voltage. Due to the non-linear cart friction behavior there is a deviation from the ideal behavior that leads to undesired oscillations mainly in state feedback based control methods. The Linear Quadratic Regulator (LQR) weight selection for the cart-inverted pendulum has been thoroughly presented. The choice of LQR is well known that unlike ordinary state feedback the LQR solution obtained after LQR weight selection automatically takes care of physical constraints. LQG compensator design also considers the white noises such as process noise and measurement noise. While designing LQG compensator a Kalman Filter was used as an optimal estimator. Lastly, Loop Transfer Recovery (LTR) analysis has been performed for suitably selecting the tuning parameter for observer design. By LTR, a set of possible tuning parameters representing the state and process noise covariances can be selected depending on the trade-off between noise suppression and system robustness.

5.2. Thesis Contributions

The following are the contributions of the thesis:

- A systematic algorithm for weight selection for LQR state feedback has been proposed and validated both in simulation and Real time.
- Kalman Filter has been designed successfully to estimate the full state vector form noisy output measurement.
- A LQG compensator has been designed and validated both in simulation and Real Time.
- Loop Transfer Recovery (LTR) is also carried out, which is also a Robust Multivariable LQG control method.

5.3. Suggestions for Future Work

A. Effect of Discretization of Control Algorithm

All the controllers designed are implemented in Real Time with the help of SIMULINK and Real-time Workshop. Since, the controller is designed on a digital platform, the effect of discretization of continuous time signals on the closed loop system performance also need to be considered in design.

B. Friction Modelling and Advanced Control Design

Friction models may also be used to identify the non-linear cart friction. This will yield a more accurate non-linear friction model. Further, the friction of the servo mechanism may also be considered. Other advanced control algorithms such as sensitivity weighted LQR and LQR with weighted cost functional, double integral sliding mode etc., may be attempted.

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